Spring 2017 MATH5012

Real Analysis II

Solution to Exercise 1

1. Let μ be a Radon measure in $\mathbb{R}^n$ and A be μ-measurable. Show that $\nu(E) = \mu(E \cap A)$ is a Radon measure.

   Solution. Straightforward.

2. Give an example showing that the condition on the uniform bound on the diameter cannot be removed in Vitali's covering theorem.

   Solution. Suffices to consider $\{B_n(0)\}$.

3. Show that for every non-empty open set G in $\mathbb{R}^n$, there is a countable, pairwise disjoint open balls B_k's in G satisfying

$$\mathcal{L}^n \left(G \setminus \bigcup_k B_k \right) = 0 \ .$$

   Solution Take $A = G$ in Corollary 6.2.

4. Show that there is no countable, pairwise disjoint open balls whose union is the open square $(0,1)^2 \subset \mathbb{R}^2$. It is known that every open set in $\mathbb{R}^1$ can be decomposed as the union of countably many disjoint open intervals. This example shows that such property no longer holds in higher dimensions.

   Solution. If the open square can be written as a countable family of open balls, take a ball and its boundary away from the boundary of the square cannot belong to another ball.

5. Give a proof of Lemma 6.5 when $\mu = \mathcal{L}^n$ and $\nu << \mathcal{L}^n$ using Corollary 6.2.

   Solution. Somewhere use the fact that μ-null set is also ν-null.

6. Let f be a Lebesgue measurable function in $\mathbb{R}^n$. Let $\mathcal{B}_x$ be the collection of all non-degenerate, closed balls touching x. Define the **maximal function** of f by

$$(Mf)(x) = \sup_{\overline{B} \in \mathcal{B}_x} \frac{1}{\mathcal{L}^n(\overline{B})} \int_{\overline{B}} |f(y)| \, dy.$$

Show that

(a) $\{x : (Mf)(x) > \alpha\}$ is open, $\forall \alpha \in [0, \infty)$.

(b) $M(f+g) \leq Mf + Mg$.

Solution:

(a) Let x be such that $(Mf)(x) > \alpha$. There exists $\overline{B} \in \mathcal{B}_x$ such that

$$\frac{1}{\mathcal{L}^n(\overline{B})} \int_{\overline{B}} |f(y)| dy > \alpha.$$

By enlarging this ball a little bit we may assume x belongs to the interior of the ball. Consequently, for all y sufficiently close to x, $B \in \mathcal{B}_y$, so

$$Mf(y) \geq \frac{1}{\mathcal{L}^n(\overline{B})} \int_{\overline{B}} |f| > \alpha \ .$$

We conclude that $\{x : \ (Mf)(x) > \alpha\}$ is open.
Note it implies that Mf is lower semicontinuous and hence Borel measurable.

(b) By the triangle inequality.

7. For $f \in L^1(\mathbb{R}^n)$, establish the following facts:

(a)

$$\mathcal{L}^n \{x : Mf(x) > \alpha\} \leq \frac{C(n) \, \|f\|_{L^1}}{\alpha} \ , \quad \alpha > 0 \ ,$$

where $C(n)$ is a dimensional constant.

(b) Mf is finite a.e.

(c) $Mf \notin L^1(\mathbb{R}^n)$ unless $f = 0$ a.e. Hint: $Mf(x) \geq \dfrac{c}{|x|^n}$, $c > 0$, $|x| \geq 1$.

Solution:

(a) Put $G = \{x : (Mf)(x) > \alpha\}$ which is open by the previous problem. For each $x \in G$, there exists some ball B such that

$$Mf(x) \geq \frac{1}{\mathcal{L}^n(\overline{B})} \int_{\overline{B}} |f(y)| dy > \alpha.$$

Let $\mathcal{F}$ denote the collection of all these balls. Observe that

$$\mathcal{L}^n(B) \leq \frac{\|f\|_{L^1}}{\alpha} \ .$$

There is a uniform bound on the diameters of these balls. By Vitali's covering theorem, we select a countable disjoint subfamily $\{B_n\}$ such that

$$G \subset \bigcup_{\mathcal{F}} \overline{B} \subset \bigcup_n \hat{B}_n \ .$$

Now

$$\begin{aligned}
\mathcal{L}^n(G) &\leq \mathcal{L}^n(\bigcup_{\mathcal{F}} \overline{B}) \\
&\leq \mathcal{L}^n(\bigcup_n \hat{B}_n) \\
&= 5^n \sum_n \mathcal{L}^n(\overline{B}_n) \\
&\leq \frac{5^n}{\alpha} \sum_n \int_{\overline{B}_n} |f| \\
&\leq \frac{5^n}{\alpha} \int_{\cup_n \overline{B}_n} |f| \\
&\leq \frac{5^n}{\alpha} \|f\|_{L^1} \ .
\end{aligned}$$

Remark. Using another version of Vitali's covering theorem, 5^n in these estimate can be replaced by 3^n, see [R1].

(b) It follows from letting $\alpha \to \infty$ in (a).

(c) WLOG suppose $\alpha \equiv \displaystyle\int_{B_1(0)} |f(y)| dy \neq 0$ (otherwise use a larger ball). For

$|x| > 1$, we have

$$\begin{aligned} Mf(x) &\geq \frac{1}{\mathcal{L}^n(\overline{B}_R(0))} \int_{\overline{B}_R(0)} |f(y)|dy \\ &\geq \frac{\alpha}{\mathcal{L}^n(\overline{B}_R(0))} \\ &\geq \frac{A}{|x|^n} \ , \quad R = |x| + 1/2 \ , \end{aligned}$$

for some constant $A > 0$ independent of x. Integrating in the polar coordinates one sees that

$$\int_B Mf(x)dx \to \infty$$

whenever B is a ball with radius tending to ∞. Hence Mf is not integrable.

8. Consider

$$\phi(x) = \begin{cases} \dfrac{1}{|x|\left(\log \frac{1}{|x|}\right)^2}, & |x| \leq \dfrac{1}{2}, \\ 0, & |x| > \dfrac{1}{2}, \ x \in \mathbb{R}. \end{cases}$$

Show that (a) $\phi \in L^1(\mathbb{R})$ and (b) $M\phi \notin L^1_{loc}(\mathbb{R})$.

Solution:

(a) We have

$$\begin{aligned} \int_{\mathbb{R}} |\phi(x)|dx &= 2\int_0^{\frac{1}{2}} \frac{1}{|x|(\log|x|)^2} dx \\ &= 2\int_{-\infty}^{-\log 2} \frac{dt}{t^2} \\ = \frac{2}{\log 2} \ . \end{aligned}$$

(b) For $0 < x < \dfrac{1}{2}$,

$$M\phi(x) \geq \frac{1}{x}\int_0^x \phi(y)dy = -\frac{1}{x\log x}.$$

Hence

$$\int_{-\delta}^{\delta} \frac{1}{|x|\log\frac{1}{|x|}} dx \geq \int_0^{\delta} \frac{1}{|x|\log 1/|x|} dx = \infty.$$

f is not even locally integrable.